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THE NAVAL SPACE COMMAND AUTOMATIC DIFFERENTIAL CORRECTION PROCESS

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The Naval Space Command maintains a database of orbital elements for the objects in the space catalog. This report is a documentation of the software which inputs new observations of a satellite and its old element set and outputs a new element set. Topics covered include: mathematics of batch least squares differential correction process, definition of fit span and passes, calculation of residuals and partials, inclusion of historical data, solution to normal equations, iterations and tolerances.

INTRODUCTION

The Naval Space Command maintains a database of element sets for roughly 9,000 Earth-orbiting objects, which is essentially the U.S. Space Command satellite catalog. NAVSPACECOM receives about 270,000 observations per day and performs an average of 18,000 element set updates per day. About 98.5% of the element sets are updated, without human intervention, by computer software called AUTODC. The purpose of our report is to elucidate the technical aspects of AUTODC for the astrodynamics community.

MATHEMATICAL BACKGROUND

AUTODC uses a batch least squares differential correction process. This is an iterative method which updates the orbital elements so that the propagated orbit fits the observations in the least squares sense, i.e., the sum of the squares of the residuals is minimized. In this section, we outline the mathematical steps in the classical method for a batch least squares differential correction of an orbital element set; for further details, see Vallado (Ref.1).

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(i) Pick an initial nominal state

$$X_{\text{nominal}} = \begin{bmatrix} X_1 \\ \vdots \\ X_8 \end{bmatrix}$$

These elements will be updated with each of the following iterations until the differential corrections are smaller than the prescribed tolerances.

(ii) Compute the values of the observed parameters Y_c at N times corresponding to the observations Y_o (assume each observation set includes 6 numbers):

$$Y_c = \begin{bmatrix} Y_{c_1} \\ \vdots \\ Y_{c_6} \end{bmatrix}_i, \ Y_o = \begin{bmatrix} Y_{o_1} \\ \vdots \\ Y_{o_6} \end{bmatrix}_i, \text{ where } i = 1, \ldots N$$

(iii) Compute the residuals or “O-Cs” $(Y_o - Y_c)_i$ (observed minus calculated parameters). These can be arranged to form the 6N×1 column matrix

$$b = \begin{bmatrix} \left(Y_{o_1} - Y_{c_1}\right)_1 \\ \left(Y_{o_2} - Y_{c_2}\right)_1 \\ \vdots \end{bmatrix}$$

(iv) Calculate the partial derivatives

$$\frac{\partial Y_c}{\partial X} = \frac{\partial Y_c}{\partial(\mathbf{r}, \mathbf{v})} \frac{\partial(\mathbf{r}, \mathbf{v})}{\partial X}$$

$$= \begin{bmatrix} \frac{\partial Y_{c_1}}{\partial r_I} & \frac{\partial Y_{c_1}}{\partial r_J} & \frac{\partial Y_{c_1}}{\partial r_K} & \frac{\partial Y_{c_1}}{\partial v_I} & \frac{\partial Y_{c_1}}{\partial v_J} & \frac{\partial Y_{c_1}}{\partial v_K} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial Y_{c_6}}{\partial r_I} & \frac{\partial Y_{c_6}}{\partial r_J} & \frac{\partial Y_{c_6}}{\partial r_K} & \frac{\partial Y_{c_6}}{\partial v_I} & \frac{\partial Y_{c_6}}{\partial v_J} & \frac{\partial Y_{c_6}}{\partial v_K} \end{bmatrix} \begin{bmatrix} \frac{\partial r_I}{\partial X_1} \cdots \frac{\partial r_I}{\partial X_8} \\ \frac{\partial r_J}{\partial X_1} \cdots \frac{\partial r_J}{\partial X_8} \\ \frac{\partial r_K}{\partial X_1} \cdots \frac{\partial r_K}{\partial X_8} \\ \frac{\partial v_I}{\partial X_1} \cdots \frac{\partial v_I}{\partial X_8} \\ \frac{\partial v_J}{\partial X_1} \cdots \frac{\partial v_J}{\partial X_8} \\ \frac{\partial v_K}{\partial X_1} \cdots \frac{\partial v_K}{\partial X_8} \end{bmatrix}$$

at each time corresponding to the observations. Here (r_I,r_J,r_K) and (v_I,v_J,v_K) denote the Cartesian coordinates of the position and velocity vectors. These can be arranged to form the 6N×8 matrix

$$A = \begin{bmatrix} \left(\dfrac{\partial Y_{c_1}}{\partial X_1}\right)_1 & \cdots & \left(\dfrac{\partial Y_{c_1}}{\partial X_8}\right)_1 \\ \left(\dfrac{\partial Y_{c_2}}{\partial X_1}\right)_1 & \cdots & \left(\dfrac{\partial Y_{c_2}}{\partial X_8}\right)_1 \\ \vdots & \vdots & \vdots \\ & & \end{bmatrix}$$

(v) Form the normal equations:

$$A^T\, W A x = A^T\, W\, b$$

Here W denotes a 6N×6N diagonal weighting matrix, and

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_8 \end{bmatrix}$$

are the differential corrections. Note that $A^T\, W A$ is an 8×8 matrix, and that $A^T\, W\, b$ is an 8×1 matrix.

(vi) Solve the normal equations:

$$x = (A^T\, W A)^{-1}\, A^T\, W\, b$$

(vii) Update the elements:

$$X_{\text{new}} = X_{\text{last}} + x$$

(For the first iteration, X_{last} is X_{nominal}.)

(viii) Apply tests to determine if iterations should continue. If so, return to step (ii).

PROGRAMMED PROCEDURE

Now we outline aspects of the differential correction process as implemented in the NAVSPACECOM software. The software has evolved from many years of experience with the maintenance of satellite orbital elements where the algorithms must be able to also handle atypical cases which arise in an operational environment. These can include cases where there is insufficient or poorly distributed observation data over the orbit of the satellite, or cases where observations have been mistagged to the incorrect satellite. The quality of the available observation data also affects how well the DC process can overcome these limitations in the data. Some of the formulas and logic used in the software were empirically derived based on experience with the satellite element set correction process.

Documentation of these methods was initiated by Perini (Ref. 2).

Units

The AUTODC software internally uses canonical units. The Earth radius $R_\oplus$ = ER and gravitational parameter $\mu_\oplus$ are taken to be unity. One canonical time unit *CTU* is then the time it takes a hypothetical satellite to travel one radian on its way around a circular orbit of radius $R_\oplus$.

To relate these units to metric values, the Earth radius and gravitational parameter are taken to be:

$$R_\oplus = 6378.135 \text{ km}, \qquad \mu_\oplus = 398597.62579588 \text{ km}^3/\text{sec}^2$$

The value for the canonical time unit is then derived from the formula

$$CTU = \sqrt{\frac{R_\oplus^3}{\mu_\oplus}} = 806.813594299989177 \text{ sec}$$

Time is measured in canonical time units from 12:00 UTC on January 1, 2000. The NAVSPACECOM inertial coordinate system is also associated with this date.

Leap seconds are not accounted for in the conversion between external (wall clock) time and internal (canonical) time units. If a leap second occurs within the fit span, the software adds/subtracts the time tick to the appropriate observation times.

The software uses radians for angle values.

Observation Types

The collection of tracking assets available to U.S. Space Command for tracking satellites non-cooperatively is known as the Space Surveillance Network (SSN). This "system of systems" presently consists of several dozen sensors, including mechanical radars, phased array radars, optical telescopes, passive radio direction-finding equipment, and the Naval Space Surveillance System. The latter system is commonly known as the Fence, is operated by Naval Space Command, and supplies unique data types discussed below. At present, AUTODC can handle the following combinations of data types from the SSN:

Azimuth and elevation angles
Right ascension and declination angles
Azimuth, elevation, and range
Earth-fixed Greenwich (EFG) Cartesian position (X,Y,Z)
Azimuth, elevation, range, plus EFG sensor location

The first three data types come from sensors whose positions are in the database. The last data type is specifically intended for mobile sensors, such as satellite-based observing platforms. Some sensors report right ascension and declination, but these angles are transformed into apparent azimuth and elevation for AUTODC processing. Some sensors can also report rates of change of azimuth, elevation, or range, but currently rate data are not used for updating cataloged orbits.

The NAVSPACECOM Fence provides, via specialized real-time interferometric processing, a pair of direction cosines of the apparent line of sight of a satellite as it passes through the Fence beam. The cosines are reckoned along nominal "East-West" and "North-South" axes at each of six receiver stations located along a great-circle arc across the southern United States. These local observation axes differ from the true geographical directions by a single rotation in azimuth, different for each station. By triangulating the apparent lines of sight from several stations, it is possible to produce an EFG Cartesian position for each satellite pass. However, this triangulated (X,Y,Z) data type is prone to systematic errors, and NAVSPACECOM uses only the single-station direction cosines for updating the satellite catalog. The real-time reduction of raw Fence data also provides estimates of the direction cosine rates, bistatic Doppler shift, and Doppler rate. The Doppler shift is used to help associate observations with known orbits.

Element Database

Several different versions of each element set are kept on the NAVSPACECOM database, for comparison purposes:

initial = first element set cataloged at NAVSPACECOM

operational = most recently updated NAVSPACECOM element set for the object

SSC = most recent element set received from U.S. Space Command for the object (SSC was acronym for Space Surveillance Center)

field = most recent element set provided to the Space Surveillance Network sensors

The operational element set update is initiated based on the arrival of new observation data into the system. If the new observation residuals calculated with the current operational element set exceed pre-established tolerances, an update of the element set is initiated. Or, if it has been more than 23 hours since the last update, an update is initiated. However, if no data is received for a satellite, an automatic update of the element set is not initiated.

The field element set update may only be initiated when NAVSPACECOM is activated for providing element sets to the sensors. Then, each field set update is initiated when the position difference with the orbit determined by the current operational set is greater than a tolerance, usually 12 km.

An element set for each orbiting object contains six classical elements augmented by two terms used to account for atmospheric drag:

$$X_{\text{classical}} = \begin{bmatrix} M \\ n \\ \dot{n}/2 \\ \ddot{n}/6 \\ e \\ \omega \\ \Omega \\ \cos i \end{bmatrix} = \begin{bmatrix} \text{mean anomaly} \\ \text{mean motion} \\ \text{first decay term} \\ \text{second decay term} \\ \text{eccentricity} \\ \text{argument of perigee} \\ \text{right ascension of ascending node} \\ \text{cosine of inclination} \end{bmatrix}$$

The second decay term is normally zero and is not corrected. Epoch is the time of the most recent observation in the last DC.

These elements are currently propagated by the subroutine PPT3. The older subroutine PPT2 and the improved subroutine PPT3 have been documented by Solomon (Ref. 3) and Schumacher and Glover (Refs. 4-5).

Fit Span and Passes

The "fit span" is the maximum length of time over which the observations are taken for a full-batch differential correction. If the satellite period is greater than or equal to 600 minutes, fit span is based on period $P = 2\pi/n$; otherwise, fit span is based on rate of change of period $\dot{P} = \left(-2\pi/n^2\right)\dot{n}$:

Period	Fit Span
$P \geq 800$ minutes	30 days
$600 \leq P < 800$ minutes	15 days
$P < 600$ minutes:	
$\dot{P} \geq -0.0005$ minutes/day	10 days
$-0.001 \leq \dot{P} < -0.0005$ minutes/day	7 days
$-0.01 \leq \dot{P} < -0.001$ minutes/day	5 days
$\dot{P} < -0.01$ minutes/day	3 days

The AUTODC software may try to expand this fit span (up to a maximum of 30 days) so as to include more observations. Note also that fit span, and many other AUTODC parameters, may alternatively be selectively adjusted by satellite rather than be set automatically by the software.

The element set is updated only if there has been at least one observation since the last DC attempt and there are more than 5 passes in the fit span. Here a "pass" is defined as:

a. The set of direction cosine observations from 2 or more receiver sites when a satellite passes through the fence beam.

b. The set of observations provided by any other SSN sensor for one satellite track over the sensor. In this case, the first observation in the fit span is the first pass. A new observation is a pass only if the time since the last observation is more than 10 minutes or if the new observation is from a different sensor.

Tolerances

If the absolute value of any O-C residual is greater than a tolerance, the corresponding single observation parameter is excluded from the iteration.

Currently the tolerances for the direction cosine, [azimuth, elevation, range], and Earth-fixed [X, Y, Z] O-Cs are all set to the same number.

The tolerances for Doppler and range-rate O-Cs currently are set to zero, which effectively excludes the corresponding observations from the differential correction process. The code for other types of rate O-Cs is not in AUTODC.

The initial tolerance is taken to be

$$\text{TOL} = 2 * \text{INT}[\max(2a - 1, 1)] \text{ nautical miles}$$

Here INT denotes real number truncation, and *a* is the mean semimajor axis in Earth radii. Some example values of the initial tolerance are as follows:

Period	Semimajor Axis	TOL	TOL
120 minutes	1.264 ER	2 nm	3.707 km
225 minutes	1.921 ER	4 nm	7.413 km
720 minutes	4.172 ER	14 nm	25.95 km
1440 minutes	6.623 ER	24 nm	44.48 km

This value for the tolerance is changed in subsequent DCs, as explained later in this report.

Residuals

Direction cosine and angular residuals and partials are multiplied by the computed range *R*; this scaling allows the use of a common tolerance with units of length for residuals of all data types. In addition, scaling factors of $1/\sin(elev)$ and $\cos(elev)$, where *elev* is elevation, are introduced in two residuals only; this decreases the importance of low-level East-West cosine observations and amplifies azimuth measurements at zenith, because O-Cs greater than the tolerance are excluded..

The differences between observed and calculated cosines are scaled as

$$OC(1) = \left(\text{East - West cosine} - \hat{\mathbf{R}} \cdot \hat{\mathbf{E}}\right) * R/\sin(elev)$$
$$OC(2) = \left(\text{North - South cosine} - \hat{\mathbf{R}} \cdot \hat{\mathbf{N}}\right) * R$$

where hats denote unit vectors. The $\hat{\mathbf{E}}$ and $\hat{\mathbf{N}}$ unit vectors lie in a plane tangent to the local horizontal, and are reckoned along and normal to the great circle arc formed by the receiver station locations.

The differences between observed and calculated azimuth and elevation are scaled as

$$OC(3) = \left[\text{azimuth} - \arctan\left(\hat{\mathbf{R}} \cdot \hat{\mathbf{E}} / \hat{\mathbf{R}} \cdot \hat{\mathbf{N}}\right)\right] * R * \cos(elev)$$
$$OC(4) = \left[\text{elevation} - \arctan(\sin(elev)/\cos(elev))\right] * R$$

For [X, Y, Z] observation data, the O-Cs are just the differences between observed and calculated positions, but expressed in the UVW coordinate frame, where $\hat{\mathbf{U}}$ is the direction from Earth center to satellite, $\hat{\mathbf{V}}$ is the transverse direction in the orbital plane, and $\hat{\mathbf{W}}$ is normal to the orbit plane.

The calculated quantities are propagated to the observation times by the PPT3 theory.

Partials of Calculated Observations

The partials of the observed parameters with respect to position and velocity are denoted in AUTODC by

$$VO = \frac{\partial Y_c}{\partial(\mathbf{r}, \mathbf{v})}$$

R times the partials of the direction cosines with respect to position **r** are

$$R^* \frac{\partial\left(\hat{\mathbf{R}} \cdot \hat{\mathbf{E}}\right)}{\partial r_j} = \hat{E}_j - \left(\hat{\mathbf{R}} \cdot \hat{\mathbf{E}}\right)\hat{R}_j$$

$$R^* \frac{\partial\left(\hat{\mathbf{R}} \cdot \hat{\mathbf{N}}\right)}{\partial r_j} = \hat{N}_j - \left(\hat{\mathbf{R}} \cdot \hat{\mathbf{N}}\right)\hat{R}_j$$

Here the subscript j = 1, 2, 3 refers to one of the Cartesian components (may be Earth-fixed or inertial coordinate frame).

R times the partials of azimuth and elevation observations with respect to **r** are

$$R^* \frac{\partial}{\partial r_j}\left[\arctan\left(\hat{\mathbf{R}} \cdot \hat{\mathbf{E}} / \hat{\mathbf{R}} \cdot \hat{\mathbf{N}}\right)\right] = \left[\left(\hat{\mathbf{R}} \cdot \hat{\mathbf{N}}\right)\hat{E}_j - \left(\hat{\mathbf{R}} \cdot \hat{\mathbf{E}}\right)\hat{N}_j\right] / \cos^2(elev)$$

$$R^* \frac{\partial}{\partial r_j}(elev) = \left[\hat{Z}_j - \left(\hat{\mathbf{R}} \cdot \hat{\mathbf{Z}}\right)\hat{R}_j\right] / \cos(elev)$$

where $\hat{\mathbf{Z}}$ is the unit vertical vector, so sin(*elev*) = $\hat{\mathbf{R}} \cdot \hat{\mathbf{Z}}$.

The partials of range with respect to **r** are

$$\frac{\partial R}{\partial r_j} = \hat{R}_j$$

For [X, Y, Z] observations, the partials with respect to **r** are

$$\frac{\partial}{\partial r_j}\left(\mathbf{r} \cdot \hat{\mathbf{U}}\right) = \hat{U}_j$$

and similarly for $\hat{\mathbf{V}}$ and $\hat{\mathbf{W}}$.

Partials of Position

The partials of the osculating position and velocity with respect to the mean elements at epoch are denoted in AUTODC by

$$PE = \frac{\partial(\mathbf{r}, \mathbf{v})}{\partial X_{\text{nonsingular}}}$$

These partials are calculated with respect to the "nonsingular" elements at epoch

$$X_{\text{nonsingular}} = \begin{bmatrix} M+\omega+\Omega \\ n \\ \dot{n}/2 \\ \ddot{n}/6 \\ e\cos(\omega+\Omega) \\ e\sin(\omega+\Omega) \\ \sin(i/2)\cos\Omega \\ \sin(i/2)\sin\Omega \end{bmatrix}$$

This choice of partials implies that the differential corrections $x_{\text{nonsingular}}$ will be in terms of nonsingular elements. The nonsingular elements, which are similar to the well-known equinoctial elements, are well-defined for zero eccentricity or zero inclination orbits; however, for $i = \pi$ they are ambiguous since Ω is undefined.

The *PE* partials are calculated from

$$PE = BR * AR = \frac{\partial(\mathbf{r},\mathbf{v})}{\partial Z} * \frac{\partial Z}{\partial X_{\text{nonsingular}}}.$$

Here *BR* denotes the 6×6 matrix of partials of the osculating (**r**, **v**) with respect to the 6 osculating elements

$$Z = \begin{bmatrix} M+\omega+\mathit{\Omega} \\ a \\ \sin(i/2)\cos\mathit{\Omega} \\ \sin(i/2)\sin\mathit{\Omega} \\ e\cos(\omega+\mathit{\Omega}) \\ e\sin(\omega+\mathit{\Omega}) \end{bmatrix}$$

and *AR* denotes the 6×8 matrix of partials of the osculating elements *Z* with respect to the 8 epoch mean elements $X_{\text{nonsingular}}$. The *AR* partials are currently based on the older PPT2 orbital theory.

Weights and Biases

The diagonal entries of the weighting matrix W currently used in AUTODC are either obtained from the database or set to a standard weight of 25×10^6.

Biases are loaded from the database and are currently set to zero for all sensors.

Formation of Normal Equations

AUTODC accumulates the effect of each set of observations in the 8×8 matrices

$$E_{\mathrm{new}} = E_{\mathrm{old}} + PE^T * VO^T * W * VO * PE$$

$$G_{\mathrm{new}} = G_{\mathrm{old}} + PE^T * VO^T * W * OC$$

(For the first set of observations, E_{old} and G_{old} are identically zero.) After iterating through all observation sets, the end result is the normal equations in the form

$$E\,x = G$$

Sequential-Batch Differential Correction

The matrix A^TWA from the previous DC is stored in the database and may be used in the current DC to represent historical data. This provides an option of performing a sequential-batch least squares differential correction with only the new observations received since the last successful DC. The mathematical derivation of the basic equations used in AUTODC is rather complicated, and significantly different from the form given by Vallado, and so is summarized here.

The least-squares method can be applied to any linear system

$$A\,x \approx b$$

where typically the m×n system has many more equations (m) than unknowns (n) and is inconsistent. Because the range of the matrix R(A) (the column space) is the orthogonal complement of the null space of the transpose N(A^T), then multiplication of the equation by the transpose always gives a consistent n×n system (though the solution may not be unique, if the variables are correlated)

$$A^T A\,x = A^T b$$

This solution minimizes the residual $r = b - A\,x$ by making it orthogonal to the column space of A, so that $A^T r = A^T (b - A\,x) = 0$. Each of the original m equations can be weighted differently using weights w_i by including a diagonal m×m matrix W with entries w_{i}^2 to give

$$A^T W A\,x = A^T W\,b$$

Then the residual r for this weighted system satisfies $A^T W\,r = A^T W\,(b - A\,x) = 0$.

For sequential batch least-squares, suppose that the first batch of data requires solving

$$A_1\,x \approx b_1$$

resulting in a (weighted) least-squares solution x_1. Then a second batch of data would require solving

$$A_2\, x \approx b_2$$

To utilize both sets of data would require solving both equations simultaneously, which corresponds to the weighted least-squares system

$$[(A^T\, W A)_1 + (A^T\, W A)_2]\; x = (A^T\, W b)_1 + (A^T\, W b)_2$$

or using the first solution x_1

$$[(A^T\, W A)_1 + (A^T\, W A)_2]\; x = (A^T\, W A)_1\; x_1 + (A^T\, W b)_2$$

This form corresponds to that given in Ref. 1. Of course, to give the older data less weight, say by a factor f, one can reduce the weights in W_1, or equivalently scale the entries of $(A^T\, W A)_1$, by a factor f^2.

In differential correction, the form above must be modified because the systems are solved for changes in the elements rather than the elements themselves, and because each batch system is iterated to account for nonlinearity, and also because each batch has a different epoch. The system to be solved for each iteration on one batch of data is

$$\frac{\partial O}{\partial X}\Delta X \approx \Delta O$$

where ΔX is the change in the elements and ΔO is the observed quantities minus the corresponding quantities calculated from the current elements (O-C's), and the coefficient matrix $\partial O/\partial X$ is the partials of the observed quantities with respect to the elements. Then after k iterations, the weighted least-squares solution for the first batch satisfies

$$\left(A^T W A\right)_1^{k-1}\left(X_1^k - X_1^{k-1}\right) = \left(A^T W\right)_1^{k-1} b_1^{k-1}$$

where the subscripts indicate the batch, superscripts indicate iteration number, X_1^k is the elements found from the first batch after k iterations, $A_1^{k-1} = \partial O_1 \big/ \partial X_1^{k-1}$, $b_1^{k-1} = \Delta O_1^{k-1} = O_1 - C_1^{k-1}$, with A_1^{k-1} and C_1^{k-1} being calculated from the previous iteration's elements X_1^{k-1}.

Then for the next batch, the epoch shifts from t_1 to t_2. The new system after n iterations is

$$A_2^{n-1}\left(X_2^n - X_2^{n-1}\right)_{t_2} \approx b_2^{n-1}$$

where the subscript t_2 indicates elements at the new epoch. To include the old data as well, the old system at the old epoch would be

$$A_1^{k-1}\left(X_2^n - X_1^{k-1}\right)_{t_1} \approx b_1^{k-1}.$$

Note that these two systems apply to the new elements X_2^n at different epochs. Small changes in elements can be shifted approximately to the old epoch using the linearization

$$\left(\Delta X\right)_{t_1} = \frac{\partial X_{t_1}}{\partial X_{t_2}}\left(\Delta X\right)_{t_2}.$$

Then the old system can be written, using Φ_{12} for $\partial X_{t_1} / \partial X_{t_2}$,

$$A_1^{k-1}\left[\Phi_{12}\left(X_2^n - X_2^{n-1}\right)_{t_2} + \left(X_2^{n-1} - X_1^k\right)_{t_1} + \left(X_1^k - X_1^{k-1}\right)_{t_1}\right] \approx b_1^{k-1}.$$

Moving the known quantities to the right-hand side and applying weighted least squares gives

$$\begin{aligned}&\Phi_{12}^T\left(A^TWA\right)_1^{k-1}\Phi_{12}\left(X_2^n - X_2^{n-1}\right)_{t_2}\\ &= \Phi_{12}^T\left(A^TW\right)_1^{k-1}\left[b_1^{k-1} - A_1^{k-1}\left(X_1^k - X_1^{k-1}\right)_{t_1} - A_1^{k-1}\left(X_2^{n-1} - X_1^k\right)_{t_1}\right]\end{aligned}$$

which simplifies, since the old elements X_1^k satisfy the old weighted least squares equation, to give

$$\Phi_{12}^T\left(A^TWA\right)_1^{k-1}\Phi_{12}\left(X_2^n - X_2^{n-1}\right)_{t_2} = \Phi_{12}^T\left(A^TWA\right)_1^{k-1}\left(X_1^k - X_2^{n-1}\right)_{t_1}.$$

Hence, combining the old data with the new gives the combined least squares system

$$\begin{aligned}&\left[\left(A^TWA\right)_2^{n-1} + \Phi_{12}^T\left(A^TWA\right)_1^{k-1}\Phi_{12}\right]\left(X_2^n - X_2^{n-1}\right)_{t_2}\\ &= \left(A^TWb\right)_2^{n-1} + \Phi_{12}^T\left(A^TWA\right)_1^{k-1}\left(X_1^k - X_2^{n-1}\right)_{t_1}\end{aligned}$$

In terms of the array names used in AUTODC, *E* is $\left(A^TWA\right)_2^{n-1}$, *G* is $\left(A^TWb\right)_2^{n-1}$, *P* is $\left(A^TWA\right)_1^{k-1}$, *EP* is Φ_{12}, and *GP* is $\left(X_1^k - X_2^{n-1}\right)_{t_1}$, so the system becomes

$$(E + EP^T\,P\,EP)\,\Delta X = (G + EP^T\,P\,GP).$$

This explains the basic equations used by AUTODC to update E and G.

Note for comparison purposes that this P is the inverse of Vallado's covariance matrix P.

In the current version of the AUTODC code, the option to perform a sequential-batch differential correction has been bypassed. The matrix A^TWA from the previous DC is set to zero, and all observations in the fit span are included in the DC process. The implementation of the historical data option is being reexamined.

Corrections in Classical Elements

If only one in a pair of classical elements, either (e, ω) or (Ω, cos i), should be corrected, the AUTODC code finds differential corrections $X_{\text{classical}}$ in terms of classical elements. For this purpose, the partials in the 8×8 Jacobian matrix are computed:

$$TX = \frac{\partial X_{\text{nonsingular}}}{\partial X_{\text{classical}}}.$$

Two of these partials contain a factor in the denominator of sin i, which is omitted in the AUTODC code, to prevent a singularity at zero inclination.

The chain rule is used to obtain the classical partials:

$$PE_{\text{classical}} = \frac{\partial(\mathbf{r},\mathbf{v})}{\partial X_{\text{classical}}} = \frac{\partial(\mathbf{r},\mathbf{v})}{\partial X_{\text{nonsingular}}} * \frac{\partial X_{\text{nonsingular}}}{\partial X_{\text{classical}}} = PE*TX$$

Thus the classical $E_{\text{classical}}$ and $G_{\text{classical}}$ matrices are:

$$E_{\text{classical}} = TX^T * E\,TX$$

$$G_{\text{classical}} = TX^T * G$$

The solution to the normal equations will be the classical correction $x_{\text{classical}}$. The chain rule is used to convert back to the nonsingular correction:

$$x_{\text{nonsingular}} = TX * x_{\text{classical}}$$

Solution to Normal Equations

The normal equations $Ex = G$ are solved by Gauss-Jordan elimination with full pivoting. The algorithm is identical to one in the book *Numerical Recipes in Fortran* by Press, et al. (Ref. 6).

The AUTODC algorithm contains the unusual feature that, when an off-diagonal element of E is too large relative to the corresponding diagonal elements, i.e. $E_{ij}^2 >$ (SING)$E_{ii}E_{jj}$, then that row (for even-number calls) or column (for odd-number calls) is inactivated. The singularity indicator SING is initially set to near one. Then, if the active matrix is singular, the threshold SING is lowered by 10% (to inactivate more rows/columns) and the solution is tried again. Eventually, either a solution is found, or the whole matrix gets made inactive (flagged by a return value SING = 0), or the threshold gets too small (SING ≤ 0.01) which indicates that the whole system is singular.

Updating Elements

Once the normal equations have been solved for the nonsingular corrections $x_{\text{nonsingular}}$, they may be added to the old elements to form the new elements. First, the classical elements $X_{\text{last classical}}$ from the last iteration are converted to the nonsingular elements $X_{\text{last nonsingular}}$.

Next, the nonsingular elements are updated:

$$X_{\text{new nonsingular}} = X_{\text{last nonsingular}} + V(1) * x_{\text{nonsingular}}$$

Here $V(1)$ is a relaxation factor used to scale down the differential corrections if any is too large. If $|x_j| > T_1(x_j)$ for any $j = 1, \ldots, 8$, the scale factor is

$$V(1) = \min_{j=1,\ldots,8}\left[\frac{T_1(x_j)}{|x_j|}\right]$$

where $T_1(x_j)$ are the numbers given below:

	$T_1(x_j)$	$T_2(x_j)$
$x_1 = M + \omega + \Omega$	5×10^{-2}	10^{-5}
$x_2 = n$	5×10^{-4}	10^{-6}
$x_3 = \dot{n}/2$	5×10^{-6}	10^{-8}
$x_4 = \ddot{n}/6$	5×10^{-8}	10^{-10}
$x_5 = e \cos(\omega + \Omega)$	5×10^{-2}	10^{-5}
$x_6 = e \sin(\omega + \Omega)$	5×10^{-2}	10^{-5}
$x_7 = e \sin(i/2) \cos \Omega$	5×10^{-2}	10^{-5}
$x_8 = e \sin(i/2) \sin \Omega$	5×10^{-2}	10^{-5}

Finally, the new nonsingular elements $X_{\text{new nonsingular}}$ are converted to the new classical elements $X_{\text{new classical}}$.

Iterations

The following tests are applied to determine if the iterations should continue:

(i) If it is the first iteration, continue iterations.

(ii) If $|x_j| > T_2(x_j)$ for any $j = 1,\ldots,8$, continue iterations. Here $T_2(x_j)$ are the numbers given above.

(iii) If $\max_{\text{last or new iteration}} \left[\frac{V(2)}{RES}\right] > .1$, continue iterations.

Here

$$RES = \sum_{i=1}^{N}\sum_{j=1}^{6}[\mathrm{WT(j)}]_i\,[OC(j)]_i^2$$

is the weighted sum of the squares of the new residuals. The parameter $V(2)$ is defined by

$$V(2) = \sum_{j=1}^{8} x_j V(1) G(j)$$

If all of the tests are met in no more than ITERI iterations, the DC is deemed to have converged. The parameter ITERI is set to 60 for the first set of iterations, and subsequently may be set to 20, or set to 0 as a flag to compute statistics.

Best Elements

AUTODC employs a scheme to iteratively tighten tolerances and repeat the element correction process in order to obtain the best element set for the given satellite. The goal is to lower the RMS while retaining an "acceptable" amount of data in the DC. The final outcome is the "best" set of mean elements. In this process, the percentage of observation data and resultant RMS used in the DC are examined for both the entire fit span and for the most recent data received since the last DC session.

An initial DC is attempted with the tolerance previously described as the initial tolerance TOL, for a maximum of ITERI = 60 iterations. If more than 85% of the observations in the fit span, and 80% of the new observations, are used, the tolerance is shrunk to

$$\mathrm{TOL} = \mathrm{STOL} * \max\left\{1, \mathrm{INT}\left[1.5\frac{\mathrm{RMS}}{\mathrm{STOL}} + .5\right]\right\}.$$

The parameter STOL is half the initial tolerance, and

$$\mathrm{RMS} = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{6}\left[OC(j)\right]_i^2 \Big/ 6N}$$

is the unweighted root mean square of the residuals. Here the RMS is calculated with either all residuals in the fit span, or just the new residuals, whichever yields the maximum RMS. Another DC iteration cycle is then attempted.

This process is repeated until the new value for TOL is less than the tolerance goal of half the initial tolerance. As the tolerance is shrunk, the DC is considered acceptable if more than 85% of the observations for the entire fit span, and 50% of the new observations, are used in the DC.

If only 75-85% of the observations in the fit span, and 50% of the new observations, were used in the last iteration cycle, no more attempts are made to shrink the tolerance. The DC is terminated and the results accepted.

During this cycle of reducing the tolerance and attempting another DC, the elements from each cycle are saved as the best element set if an acceptable amount of data was retained in the DC. If this process of reducing the tolerance does not converge to the tolerance goal with an acceptable amount of data, the process is terminated and the best element set is accepted as the new element set.

Provisions are made in AUTODC if the initial tolerance is too low for an acceptable DC on the first attempt. In this case, the tolerance is changed to TOL = 1 ER and a DC is performed with zero iterations to compute statistical information only. Then a third DC is performed with ITERI = 20 and

$$\mathrm{TOL} = \mathrm{STOL} * \mathrm{INT}\left\{3\frac{\max\left[\mathrm{STOL}, \mathrm{RMS}\right]}{\mathrm{STOL}} + 1\right\}.$$

The process then continues, as described above, to find the best element set.

CONCLUSION

This has been a brief summary of some of the salient features of the NAVSPACECOM differential correction process. For more explanation, and for the Fortran source code, the reader is referred to the documentation of Danielson and Canright (Ref. 7).

Now that this documentation has been completed the next step is to study where improvements can be made in the software. Progress in this direction has been made by

Marshall (Ref. 8), who showed the benefits of Singular Value Decomposition when solving the normal equations. Goals of the improvement process include

(i) Enabling the software to update 100% of the element sets automatically

(ii) Improving the accuracy of the state and covariance

(iii) Decreasing computer run time.

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